

THE CLASSIFICATION PROBLEM FOR GRAPHS AND LATTICES IS WILD

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ABSTRACT. We prove that the classification problem for graphs and several types of algebraic lattices (distributive, congruence and modular) up to isomorphism contains the classification problem for pairs of matrices up to simultaneous similarity.

1. INTRODUCTION

The graph isomorphism problem is one of the central problems in graph theory. Reducing this problem to the isomorphism problem for algebraic structures, such as rings, algebras and groups, was studied in several works, e.g., [Kim, Roush '80], [Droms '87] and [Saxena, Agrawal '05]. The classification problem for finite and infinite algebraic lattices has also been extensively addressed. A wild classification problem contains the problem of classification of pairs of matrices up to simultaneous similarity. In this paper, we prove that the classification problem for graphs is wild by reducing the classification problem for finite 2-nilpotent p -groups to the classification problem for graphs (the wildness of classification problem for finite 2-nilpotent p -groups was proved in [Sergeichuk '75]). We use wildness of the classification problem for graphs to show that the classification problem for algebraic lattices and poset lattices is wild. A reduction from graphs to lattices, described in this paper, allows us to prove that the classification problem for distributive, modular and congruence lattices is wild, even for finite lattices.

2. WILDNESS

We use in this paper the following definitions of a matrix problem and wildness, first given in [Belitskii, Sergeichuk '03]. A *matrix problem* $\{\mathcal{A}_1, \mathcal{A}_2\}$ is a pair that consists of a set \mathcal{A}_1 of a -tuples of matrices from $M_{n \times m}$, and a set \mathcal{A}_2 of admissible matrix transformations. Given two matrix problems $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$, we say that the matrix problem \mathcal{A} is *contained in* the matrix problem \mathcal{B} if there exists a b -tuple $\mathcal{T}(x) = \mathcal{T}(x_1, \dots, x_a)$ of matrices, whose entries are non-commutative polynomials in x_1, \dots, x_a , such that

$$(1) \quad \mathcal{T}(A) = \mathcal{T}(A_1, \dots, A_a) \in \mathcal{B}_1 \text{ if } A = (A_1, \dots, A_a) \in \mathcal{A}_1;$$

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- (2) for every $A, A' \in \mathcal{A}_1$, A reduces to A' by transformations \mathcal{A}_2 if and only if $\mathcal{T}(A)$ reduces to $\mathcal{T}(A')$ by transformations \mathcal{B}_2 .

In a *pair of matrices* matrix problem, denoted by $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$, we have

$$\mathcal{W}_1 = \{A, B \mid A, B \in M_{n \times n}\}$$

and

$$\mathcal{W}_2 = \{S(A, B)S^{-1} \mid S \in M_{n \times n} \text{ non-singular}\}.$$

A matrix problem is called *wild* if it contains \mathcal{W} , and *tame* otherwise.

3. CLASSIFICATION OF GRAPHS

The main result we use in this section is the wildness of classification problem for finite 2-nilpotent p -groups, which was established in [Sergeichuk '75]. The classifying problem for finite 2-nilpotent p -groups contains a problem of reducing skew-symmetric matrices over $\mathbb{Z}/p\mathbb{Z}$ by congruence transformations to block-triangle matrices. The latter problem is a matrix problem and it contains the pair of matrices problem \mathcal{W} . We formulate this theorem here for completeness.

Theorem 3.1. [Sergeichuk '75] *Let G be a 2-nilpotent finite p -group which is an extension of an abelian group A by an abelian group B :*

$$1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1.$$

The problem of classifying of such groups G with group A of the order p is tame. However, if the order of A is more than p , the above problem is wild.

It remains to show that the classification problem for finite 2-nilpotent p -groups can be reduced to the classification problem for graphs. We prove here a stronger result, reducing the classification problem for finite groups to the classification problem for graphs. We do it by constructing a graph corresponding to a given finite group so that the resulting graphs are isomorphic if and only if their source groups are isomorphic.

Let $G = (A, \circ)$ be a finite group. We construct a directed edge-colored graph $\Gamma(G) = (V, E)$ corresponding to G as follows. The node set of $\Gamma(G)$ is the union of A and all the ordered triples from $A \times A \times A$. For every triple $u, v, w \in A$ such that $u \circ v = w$, the edge set of $\Gamma(G)$ contains the edge $(u, (u, v, w))$ of color 1, the edge $(v, (u, v, w))$ of color 2 and the edge $((u, v, w), w)$ of color 3. For the sake of completeness, we prove here the following theorem (see, e.g., [Hoffman '81]).

Theorem 3.2. *Let $G = (A, \circ)$ and $H = (B, \cdot)$ be finite groups. Then $G \approx H$ if and only if $\Gamma(G) \approx \Gamma(H)$.*

Proof. Let $\Gamma(G) = (V, E)$ and $\Gamma(H) = (V', E')$. The “only if” direction is trivial, since every isomorphism ϕ from G to H can be extended to a graph isomorphism $\psi : V \rightarrow V'$ so that $\psi(a) = \phi(a)$ for $a \in A$ and $\psi(a, b, c) = (\phi(a), \phi(b), \phi(c))$ for $(a, b, c) \in A \times A \times A$.

Suppose now that $\psi : V \rightarrow V'$ is a graph isomorphism. Since ψ preserves node in-degrees and node out-degrees, it maps A to B and $A \times A \times A$ to $B \times B \times B$. Therefore, the restriction of ψ to A , denoted by ψ_A , is a bijection from A to B . It remains to show that ψ_A preserves the group operation. Let $u, v, w \in A$ so that $u \circ v = w$. Since ψ is a graph isomorphism, $(u, (u, v, w)), (v, (u, v, w)), ((u, v, w), w) \in E$. As ψ preserves edge colors, the edges $(\psi(u), \psi(u, v, w)), (\psi(v), \psi(u, v, w))$ and $(\psi(u, v, w), \psi(w))$ lie in E' and have colors 1, 2 and 3 correspondingly. Then we have $\psi_A(u) \cdot \psi_A(v) = \psi_A(w)$ and ψ_A therefore is a group isomorphism. \square

Corollary 3.3. *The classification problem for graphs is wild.* \square

4. CLASSIFICATION OF LATTICES

Let algebra $\langle L, \wedge, \vee \rangle$ be a lattice. Then L is equivalent to a poset lattice $\mathcal{L} = \langle L, \leq \rangle$ where $a \leq b$ if and only if $a \wedge b = a$. This equivalence ensures that for any two lattices L_1, L_2 and their corresponding poset lattices $\mathcal{L}_1, \mathcal{L}_2$, it holds that $L_1 \cong L_2$ if and only if $\mathcal{L}_1 \cong \mathcal{L}_2$. Likewise, given a poset lattice $\mathcal{L} = \langle L, \leq \rangle$ one can define a corresponding algebraic lattice $\langle L, \wedge, \vee, \rangle$ by setting $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ for all $a, b \in L$. In this case, $\mathcal{L}_1 \cong \mathcal{L}_2$ if and only if $L_1 \cong L_2$ (see [Grätzer 1971]). Therefore, if one wishes to show the wildness of the classification problem for lattices and poset lattices, it is enough to prove the wildness of one class of the two.

Theorem 4.1. *The classification problem for lattices is wild.*

Proof. We prove the theorem by constructing a simple isomorphism-preserving reduction from undirected graphs to poset lattices. Let $G = (V, E)$ be an undirected graph. We construct a directed incidence graph $G_1 = (V_1, E_1)$ corresponding to G by setting $V_1 = V \cup (V \times V)$ and let E_1 contain the edges $(\{a, b\}, a)$ and $(\{a, b\}, b)$ whenever $\{a, b\} \in E$. Trivially, for any two undirected graphs G and F and their corresponding incidence graphs G_1 and F_1 we have $G \cong F$ if and only if $G_1 \cong F_1$. The incidence structure of G_1 is already a poset (see Figure 1(a)) but is not yet a poset lattice. For this purpose, we extend the graph G_1 to an *extended incidence graph* $G_2 = (V_2, E_2)$ where $V_1 \subset V_2$ and $E_1 \subset E_2$ and $V_2 = \{v_1, \dots, v_n\} \cup \{a_1, \dots, a_n\} \cup \{b_{1,1}, \dots, b_{n,n}\} \cup \{\text{Inf}\} \cup \{\text{Sup}\}$. The edge set E_2 contains, in addition to E_1 , the following edges.

- (1) $(\text{Sup}, b_{i,j})$ for all $1 \leq i, j \leq n$,
- (2) (a_i, Inf) for all $1 \leq i \leq n$,
- (3) $(b_{i,j}, \{v_i, v_j\})$ for all $1 \leq i, j \leq n$ and
- (4) (v_i, a_i) for all $1 \leq i \leq n$.

The incidence structure of G_2 , shown in Figure 1(b), is a poset lattice. A trivial topological order argument shows that for any two undirected graphs G and F and their corresponding extended incidence graphs G_2 and F_2 , $G \cong F$ if and only if $G_1 \cong F_1$. Since the incidence structure of an extended incidence graph is a poset lattice, we have a (polynomial) reduction from graphs to finite poset lattices. Since algebraic and poset lattices are equivalent, and the

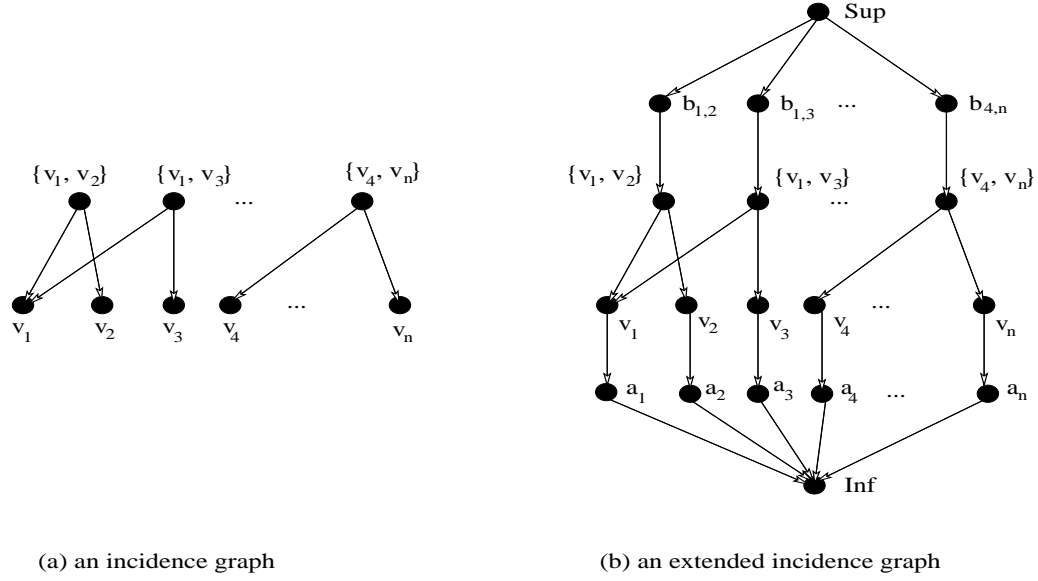


FIGURE 1. An incidence graph and an extended incidence graph.

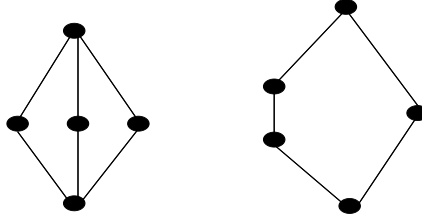


FIGURE 2. The diamond lattice M_3 (left) and the pentagon lattice N_5 (right).

classification problem for graphs is wild, then so is the classification problem for algebraic and poset lattices. \square

A lattice $\langle L, \wedge, \vee \rangle$ is called *distributive* if each three elements $x, y, z \in L$ satisfy

$$(1) \quad \begin{aligned} (x \wedge y) \vee (x \wedge z) &= x \wedge (y \vee z) \\ (x \vee y) \wedge (x \vee z) &= x \vee (y \wedge z) \end{aligned}$$

A lattice $\langle L, \wedge, \vee \rangle$ is *modular* if it satisfies $x \leq b \Rightarrow x \vee (a \wedge b) = (x \wedge a) \vee b$ for all $x, a, b \in L$. Distributive lattices are characterized as follows: a lattice is distributive if and only if none of its sublattices is isomorphic to the diamond lattice M_3 or the pentagon lattice N_5 (see Figure 2). Since a lattice is modular if and only if it does not contains N_5 (see [Grätzer 1971]), every distributive lattice is also modular.

Theorem 4.2. *The classification problem for distributive and modular lattices is wild.*

Proof. It is easy to show that a lattice defined by an extended incidence graph corresponding to an undirected graph of Theorem 4.1 is distributive and is therefore modular. Indeed, suppose that an extended incidence graph lattice contains the diamond lattice M_3 . Let us denote this extended incidence graph by G_2 ; the names of the nodes of G_2 are assumed to be as defined above (see Figure 1(b)). Then a node of G_2 corresponding to the top element of M_3 must have an out-degree 3 and can therefore only be the node Sup. Then the node corresponding to the bottom element of M_3 must have an in-degree 3 and be one of the nodes $\{v_i, v_j\}$, which all have in-degree 1 – a contradiction. Let us assume now that a lattice corresponding to G_2 contains the pentagon N_5 as a sublattice. Since the top element of N_5 has an out-degree 2, it may only correspond to the nodes Sup or $\{v_i, v_j\}$ of G_2 . In the former case, any directed path of length 2 beginning in Sup ends in a node $\{v_i, v_j\}$, while all directed paths of length 3 beginning in the node Sup must end in one of the nodes v_i . Thus the two paths cannot have a common end. In the latter case, a directed path of length 3 starting in $\{v_i, v_j\}$ must end in the Inf node, while a directed path of length 2 starting in $\{v_i, v_j\}$ has to end in one of the a_k nodes; those two paths cannot therefore have a common end. Therefore, G_2 (as a poset lattice) does not contain N_5 as a sublattice. \square

A lattice L is a *congruence lattice* if there exists an algebra A so that the lattice of all congruences of A under inclusion is isomorphic to L . We use the following result of T. Katriňák that was proved in [Katriňák 1994].

Theorem 4.3 (T. Katriňák, 1994). *Every finite distributive lattice D is isomorphic to the congruence lattice of a finite p -algebra P .*

Then Theorem 4.2 implies the following.

Corollary 4.4. *The classification problem for congruence lattices is wild.*

Proof. Distributive lattices of Theorem 4.2 are finite. \square

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